OVERVIEW OF BEAM INSTABILITIES

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CONTENT

1) Introduction:

The motion of a single particle in a ring is determined by external guide fields (dipole and quadrupole magnets, RF-system) and initial conditions. The many particles in a high intensity beam represent them self charges and currents which are sources of electromagnetic fields (self fields). They are modified by the boundary conditions (**impedance**) of the beam surroundings (vacuum chambers, cavities, etc.) and act back on the beam. This can lead to a **frequency shift** (change of the betatron or synchrotron frequency), to an increase of a small perturbation of the beam, i.e. an **instability** or to a **change of the particle distribution**, e.g. bunch lengthening.

If the self-fields are small compared to the guide field and their effect is treated as a perturbation. In some cases, like bunch lengthening, a self-consistent distribution has to be found.

Multi-traversal effects require an impedance with memory, usually a narrow band cavity. **Single traversal** effects can be driven by a broad-band impedance. The two effects are usually treated separately, except for **continuous (unbunched, coasting)** beams. These instabilities can be **longitudinal** or **transverse**, involving synchrotron or betatron oscillations.

2) Longitudinal coupled bunch instabilities Robinson instability

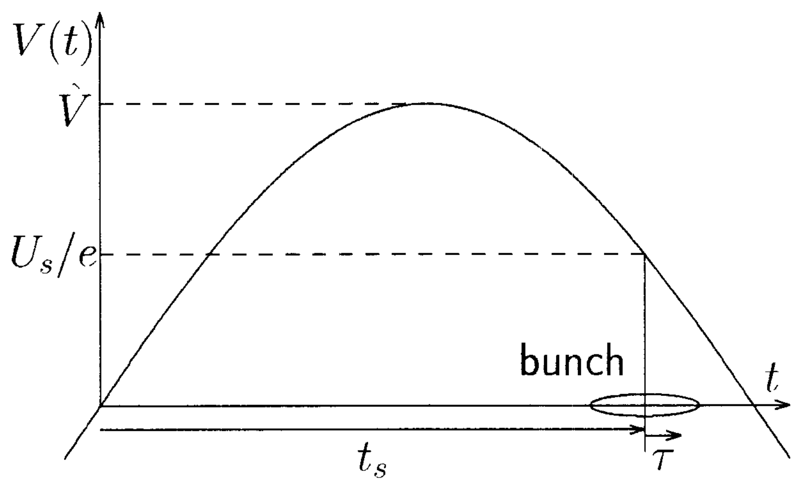
The longitudinal dynamics is based on the relation between the deviations from nominal energy E and revolution frequency ω_0 expressed by the momentum compaction α_c.

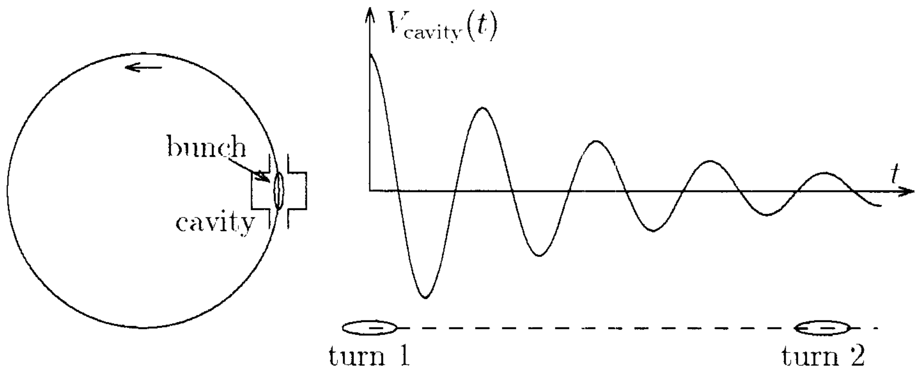
$$\frac{\Delta E}{E} = \beta^2 \frac{\Delta p}{p} = -\frac{\beta^2}{\eta_c} \frac{\Delta \omega_0}{\omega_0}, \text{with } \eta_c = \alpha_c - \frac{1}{\gamma^2}.$$

The RF-cavity oscillates with harmonic frequency $\omega_{RF} = h\omega_0$ giving a particle arriving at the synchronous time t_s or phase $\phi_s = \omega_{RF} t_s$ an energy $U_s = e\hat{V} \sin\phi_s$ per turn to compensate losses. Particles arriving before or after t_s receive a different energy. This, together with the energy dependence of ω_0, leads to longitudinal focusing. Particles execute synchrotron or energy oscillations with frequency ω_s

$$V_{RF}(t) = \hat{V} \sin(h\omega_0 t)\ ,\quad \omega_s^2 = -\omega_0^2 \frac{\eta_c h e \hat{V} \cos\phi_s}{2\pi\beta^2 E} = Q_s^2 \omega_0^2.$$

Above transition energy $\eta_c > 0$ and $\cos\phi_s < 0$,
below transition energy $\eta_c < 0$ and $\cos\phi_s > 0$.

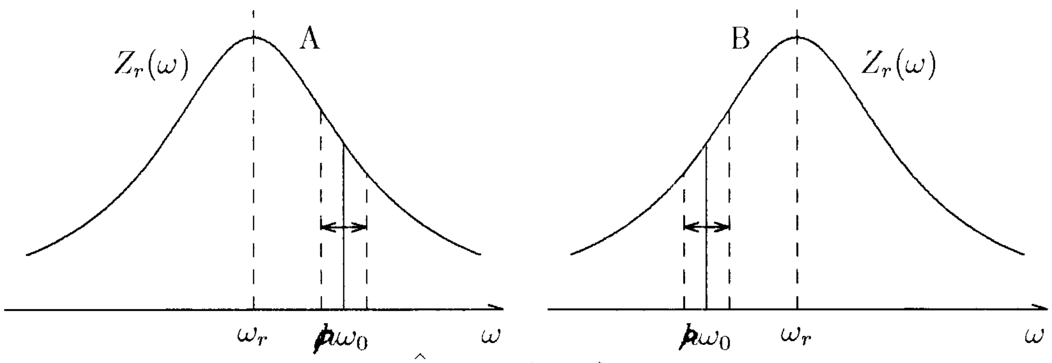




A bunch circulating in a storage ring induces a voltage in a cavity which can change its energy on the next turn. This can increase a synchrotron oscillation and lead to an instability. We consider a single, narrow cavity resonance with shunt impedance R_s and frequency $\omega_r \approx p\omega_0$. The current component I_p of the bunch at the frequency $\omega_p = p\omega_0$ induces a voltage

$$V(t) = I_p\hat{V}\left(Z_r\cos(p\omega_0 t) - Z_i\sin(p\omega_0 t)\right) = I_p R_s \frac{\cos(\omega_p t) - Q\frac{\omega_r^2-\omega_p^2}{\omega_r\omega_p}\sin(\omega_p t)}{1 + Q^2\left(\frac{\omega_r^2-\omega_p^2}{\omega_r\omega_p}\right)^2}$$

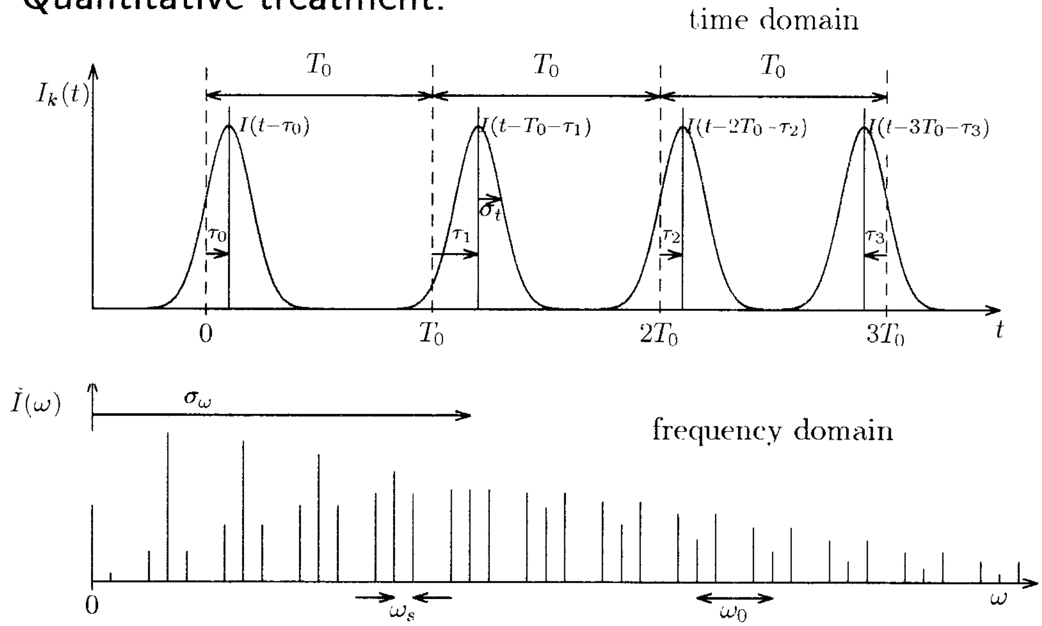
Qualitative treatment:



Oscillating bunch $\Delta E = \hat{\Delta E}\cos(\omega_s t)$, above transition, $\eta_c > 0$

A) Cavity tuned to $\omega_r < p\omega_0$.
If $\Delta E > 0$, $\Delta\omega_0 < 0$, impedance large, energy loss in cavity large.
if $\Delta E < 0$, $\Delta\omega_0 > 0$, impedance small, energy loss in cavity small.
This condition is stable.
B) Cavity tuned to $\omega_r > p\omega_0$.
If $\Delta E > 0$, $\Delta\omega_0 < 0$, impedance small, energy loss in cavity small.
if $\Delta E < 0$, $\Delta\omega_0 > 0$, impedance large, energy loss in cavity large.
This condition is instable.
Below transition energy the situation is reversed.

Quantitative treatment:



The oscillating bunch creates side-bands to the revolution harmonics at the frequencies and with current components

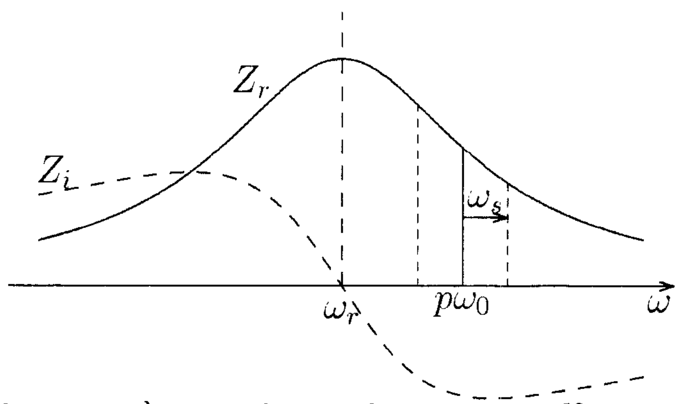
$$\omega_{p\pm} = \omega_0(p \pm Q_s), \quad I_{p\pm} = \frac{\omega_0}{\sqrt{2\pi}}\tilde{I}(\omega_{p\pm}), \quad \tilde{I}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} I(t)\mathrm{e}^{-\omega t}dt.$$

Including the voltage induced at $\omega_{p\pm}$ in Z the longitudinal dynamics gives a synchrotron oscillation with an exponential growth (or damping) rate and frequency shift

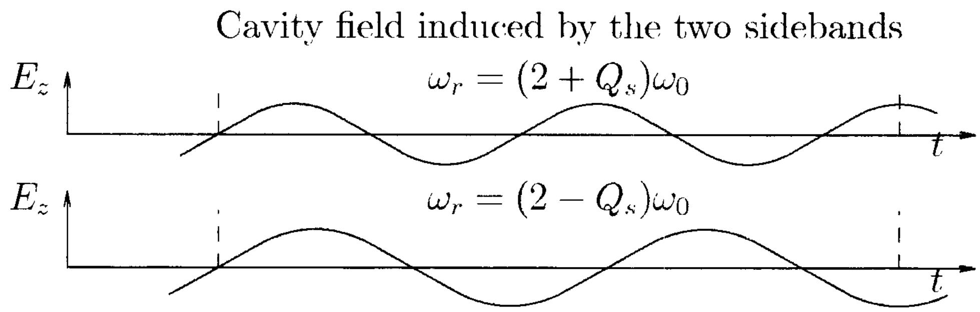
$$\Delta E = \Delta\hat{E}\mathrm{e}^{-\Delta\omega_{si}t}\cos\left((\omega_s + \Delta\omega_{sr})t\right)$$

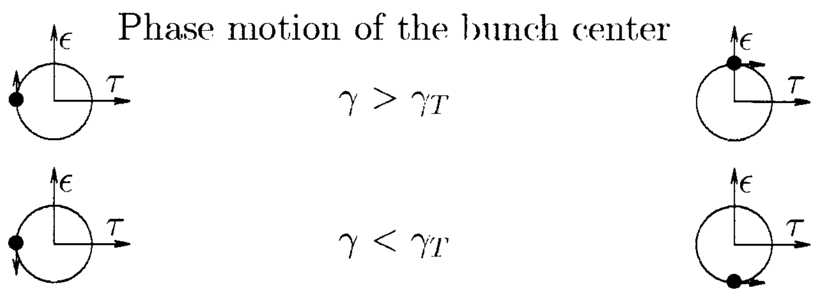
$$\frac{1}{\tau_s} = \Delta\omega_{si} = \omega_s \frac{\Big((p+Q_s)I_{p+}^2 Z_r(\omega_{p+}) - (p-Q_s)I_{p-}^2 Z_r(\omega_{p-})\Big)}{8hI_0\hat{V}\cos\phi_s}$$

$$\Delta\omega_{sr} = \omega_s \frac{\Big((p+Q_s)I_{p+}^2 Z_i(\omega_{p+}) + (p-Q_s)I_{p-}^2 Z_i(\omega_{p-})\Big)}{8hI_0\hat{V}\cos\phi_s}$$



The growth (or damping) rate depends on the difference in impedance between the upper and lower sideband.

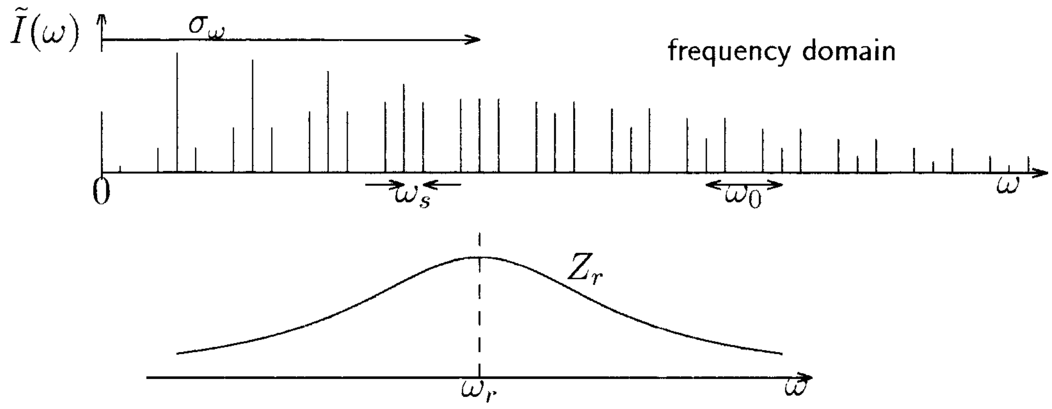




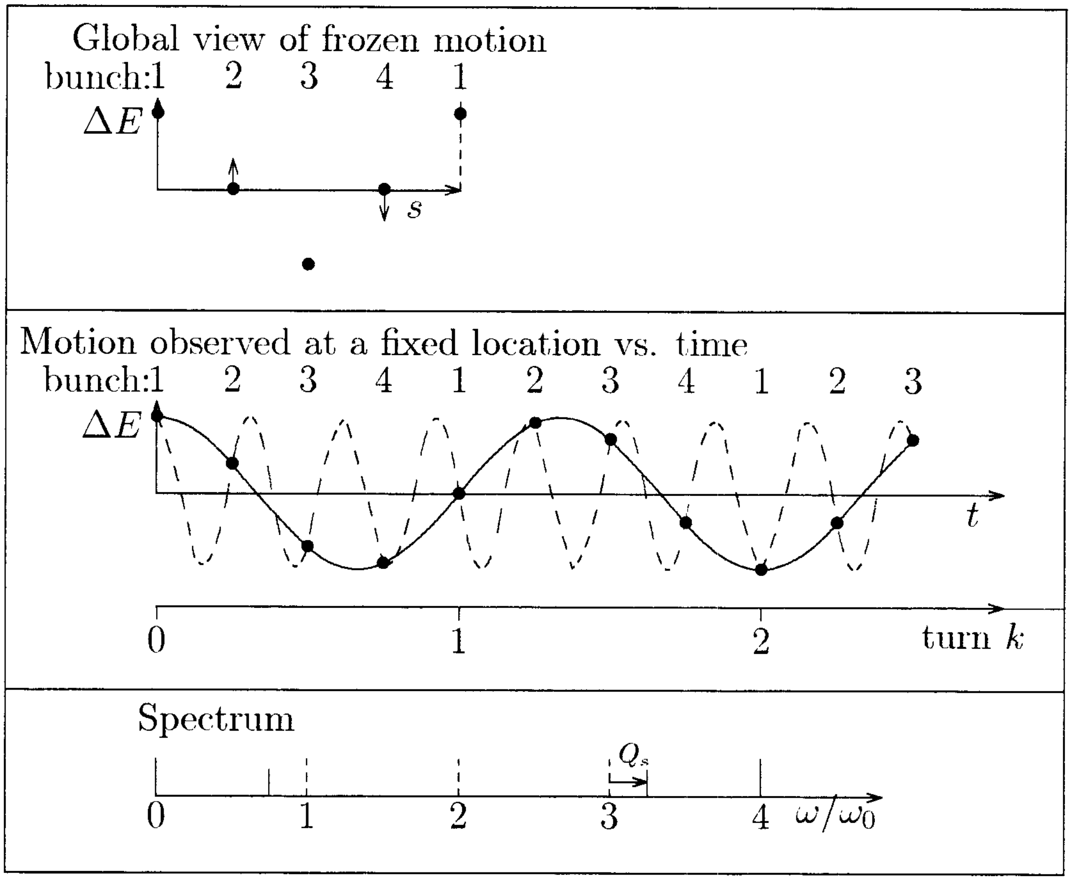
This can be understood qualitatively from the above figure having $p = 2$ and $Q_s = \omega_s/\omega_0 = 0.25$. At $\gamma > \gamma_T$ the voltage induced by the upper sideband enhances the oscillation, the one from the lower sideband reduces it. Below transition the situation is reversed.

General impedance: We have to take the impedance at the sidebands of all revolution harmonics $p\omega_0$

$$\frac{1}{\tau_s} = \frac{\omega_s}{8hI_0\hat{V}\cos\phi_s}\sum_p \Big((p+Q_s)I_{p+}^2(Z_r(\omega_{p+}) - (p-Q_s)I_{p-}^2 Z_r(\omega_{p-})\Big)$$



Complex impedance: The expressions become more compact using $Z(\omega) = Z_r(\omega) + jZ_i(\omega)$ and $\Delta\omega = \Delta\omega_{sr} + j\Delta\omega_{si}$ with $\exp(j\omega t)$ using positive and negative frequencies.



Many bunches: With M equidistant bunches in the ring there are M independent modes of coupled bunch oscillations labeled by the coupled mode number $0 \leq n \leq M-1$ which is related to the difference $\Delta\phi$ of oscillation phase between adjacent bunches $n = \Delta\phi/(2\pi M)$. Each mode n has one pair of sidebands in each frequency range of $M\omega_0$ at

$$\omega_{p\pm} = \omega_0(pM \pm (n + Q_s))$$

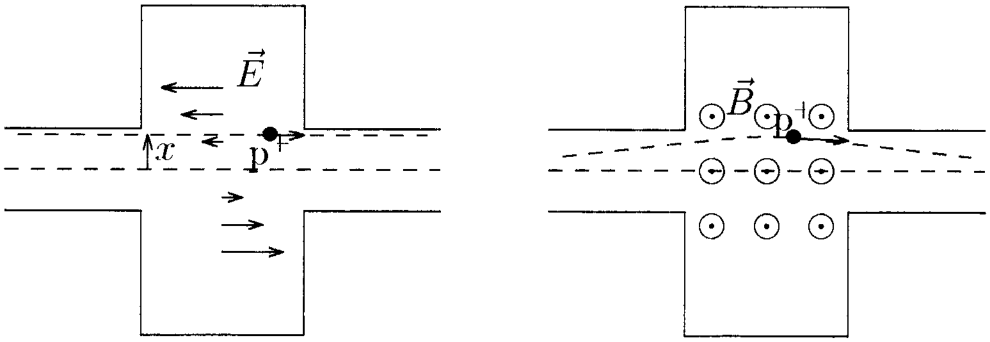
The above figure shows an example for $M = 4$, $n = 1$ and $Q_s = \omega_s/\omega_0 = 0.25$ The growth rate of each mode n is given by a sum over the impedance differences of each sideband pair.

$$\frac{1}{\tau_s} = \frac{\omega_s}{8hI_0\hat{V}\cos\phi_s} \sum_p \Big((p+Q_s)I_{p+}^2(Z_r(\omega_{p+}) - (p-Q_s)I_{p-}^2 Z_r(\omega_{p-})\Big)$$

Bunch shape oscillations: In addition to the rigid dipole modes ($m = 1$) there are bunch shape oscillations, quadrupole mode ($m = 2$), sextupole mode ($m = 3$), ... with the frequencies

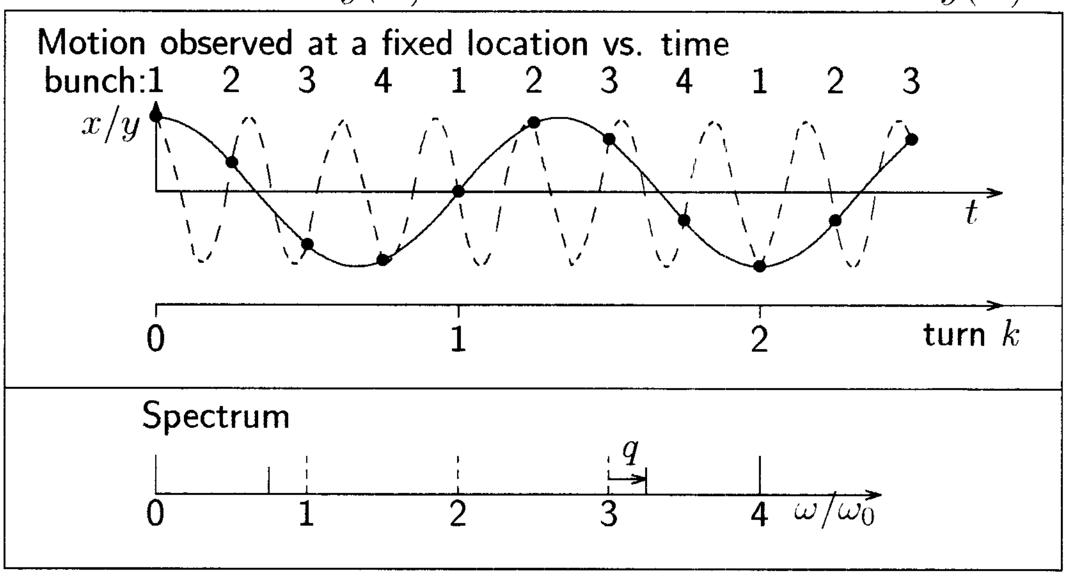
$$\omega_{p\pm} = \omega_0(pM \pm (n + mQ_s)).$$

3) Transverse coupled bunch instabilities



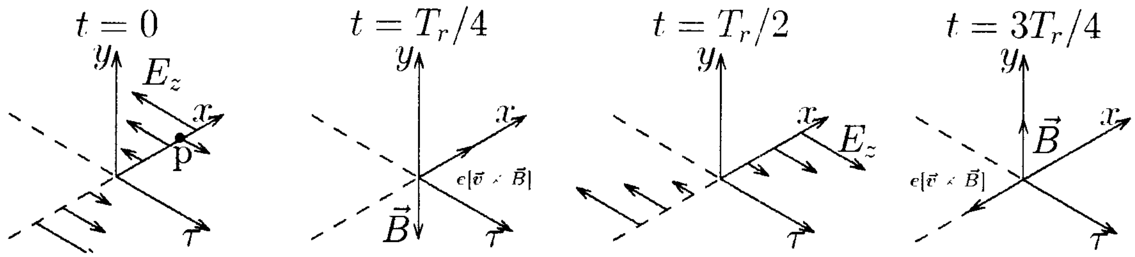
Transverse impedance: Cavities can have transversely deflecting modes. They are excited by the beam through the longitudinal electric field which is converted into a deflecting magnetic field a quarter of an oscillation period $T_r/4$ later. The impedance is given by an integral around the ring over the Fourier component of the deflecting fields divided by the Fourier component of the dipole moment. The $'j'$ indicates that deflection and dipole moment are out of phase which is avoided if we take the time derivative of the latter

$$Z_T(\omega) = j\frac{\oint\left(\vec{E}(\omega) + [\vec{\beta}c \times \vec{B}(\omega)]\right)_T}{Iy(\omega)} = -\frac{\omega\oint\left(\vec{E}(\omega) + [\vec{\beta} \times \vec{B}(\omega)]\right)_T ds}{I\dot{y}(\omega)}$$



A single bunch executing a betatron oscillation with tune $Q =$ integer $+\, q$ excites a cavity with a frequencies $\omega_0(p \pm q)$. For M equidistant bunches there are M coupled modes possible labeled with $0 \leq n \leq M-1$ with frequencies

$$\omega_{p\pm} = \omega_0\left(pM \pm (n+q)\right)$$

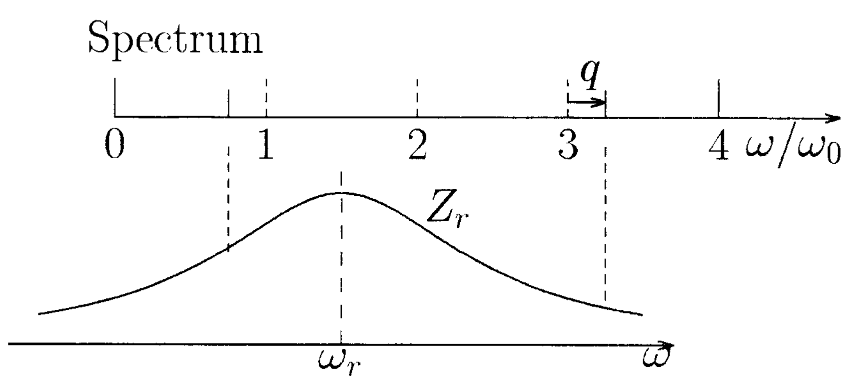


A bunch passes with a displacement x through the cavity and excites a fields $\vec{E}$ which converts after $T_r/4$ into a field $-\vec{B}$, then into $-\vec{E}$ and after into $\vec{B}$. The oscillating bunch has sidebands at $\omega_0(\text{integer} \pm q)$ (we take $q = 1/4$). With the cavity tuned to the upper sideband the bunch will traverse it in the next turn at $t = T_r(k + 1/4)$ with a transverse velocity in the $-x$ direction and receive by the magnetic field a force in the opposite direction which damps the oscillation. With the cavity tuned to the lower sideband the bunch traverses at $t = T_r(k' + 3/4)$ with a negative velocity and receive a force in the same direction. Like in the longitudinal case the resistive impedances at the two sidebands have opposite effects leading to a similar expression for the growth or damping rate.

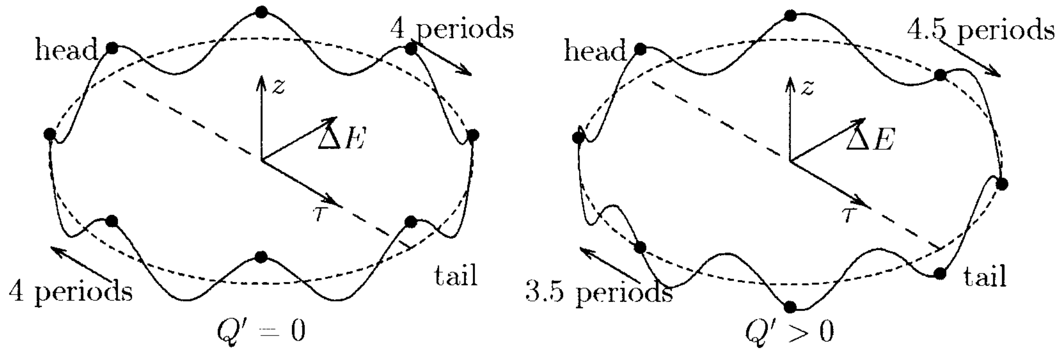
$$\frac{1}{\tau_s} \propto \sum_p \left(I_{p+}^2 (Z_T(\omega_{p+}) - I_{p-}^2 Z_T(\omega_{p-}) \right)$$

with

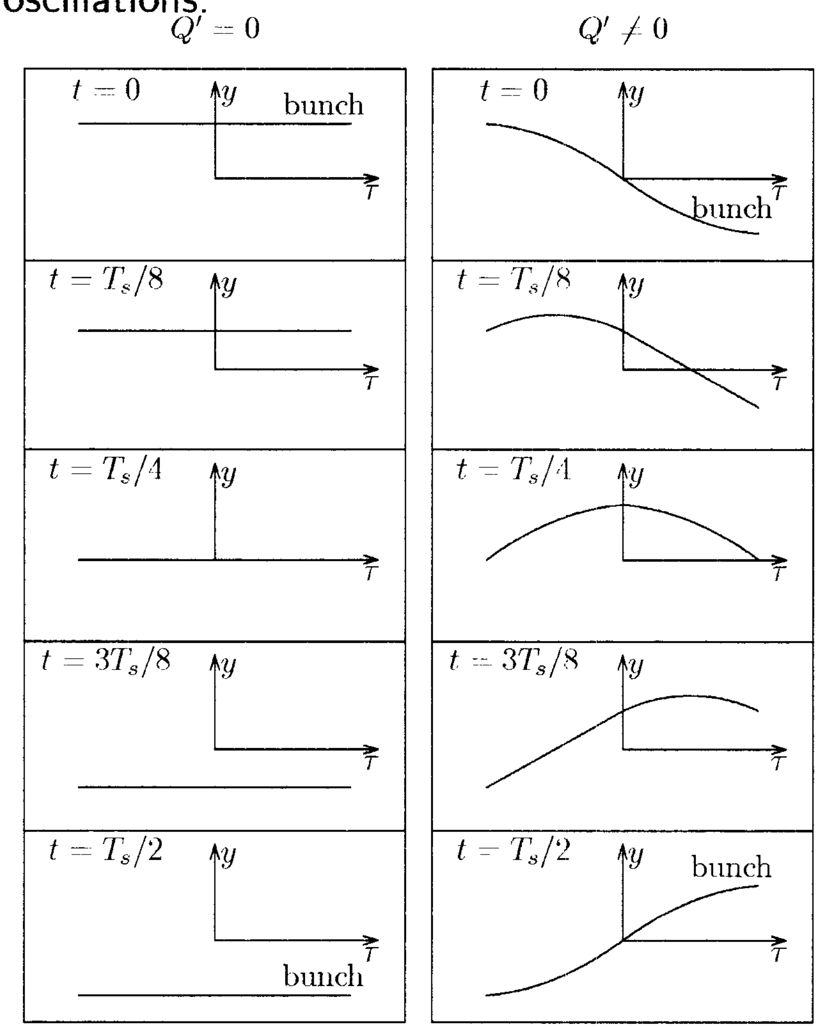
$$\omega_{p\pm} = \omega_0 \left(pM \pm (n + q) \right).$$



4) Single traversal head-tail instability



The transverse head-tail instability involves coherent transverse bunch motion and incoherent longitudinal motion. The synchrotron oscillations of a particle represents an ellipse in the phase space of energy ΔE and time τ deviation. A simultaneous transverse oscillation can be influence by this through the chromaticity $Q' = dQ/(dp/p)$. A particle going from head to tail of the bunch due to a synchrotron oscillation has for $(\gamma_T > 0)$ an excess energy and, for $Q' > 0$, a higher betatron frequency. This is reversed in the second half of the oscillation, going from tail to head. Consequently there is a betatron phase advance between head and tail for the coherent transverse oscillations.

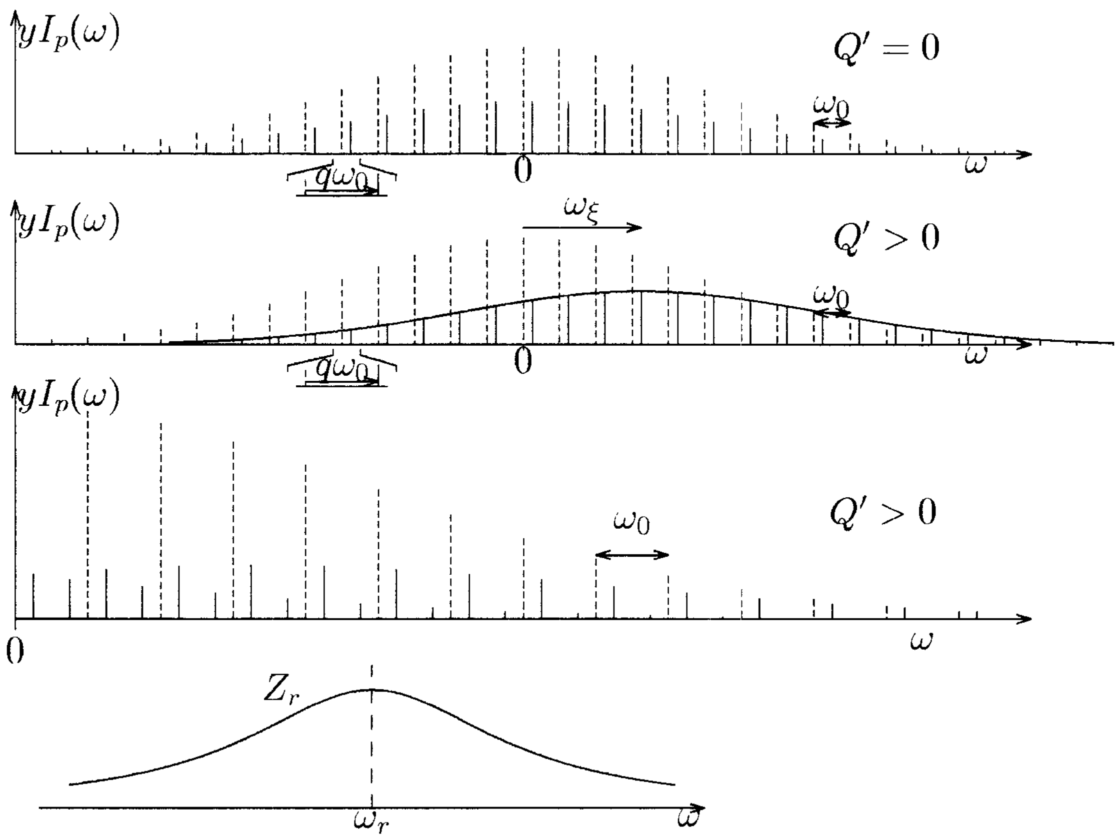


Instability: A broad band impedance, with a memory as short as the bunch length, can be exited by an oscillation of the particles A at the bunch head. This can in turn excite an oscillation of the particles B at the tail with a certain phase $\Delta\phi$ compared to the head. After half a synchrotron oscillation period the particles B are at the head and the particles A at the tail oscillating with a phase $-\Delta\phi$ compared to B assuming $Q' = 0$. The excitation by the head has now the wrong phase to keep the oscillation growing unless the chromaticity is finite and has the correct sign. In this case a head-tail instability can occur.

The 'wiggle' of the head-tail motion shifts the envelope of the side-bands by $\omega_\xi = Q'\omega_0/\eta_c$ and we have current components

$$I_{p\xi\pm} = \frac{\omega_0}{\sqrt{2\pi}}\tilde{I}(\omega_{p\pm} \pm \omega_\xi), \;\; \omega_{p\pm} = \omega_0\,(pM \pm (n+q))$$

which can be very different for the two adjacent sidebands. Even a very broad band impedance can lead to an instability.



5) Coasting beam instabilities

Transverse: The transverse betatron frequencies of a beam with nominal momentum are

$$\omega_{\beta f} = (n_f + Q)\omega_0 \ \ , \ \ \omega_{\beta s} = (n_s - Q)\omega_0.$$

Through

$$\frac{\Delta E}{E} = \beta^2 \frac{\Delta p}{p} = -\frac{\beta^2}{\eta_c}\frac{\Delta \omega_0}{\omega_0} \text{ , and } \ \Delta Q = Q' \frac{\Delta p}{p}.$$

they are affected by a momentum deviation

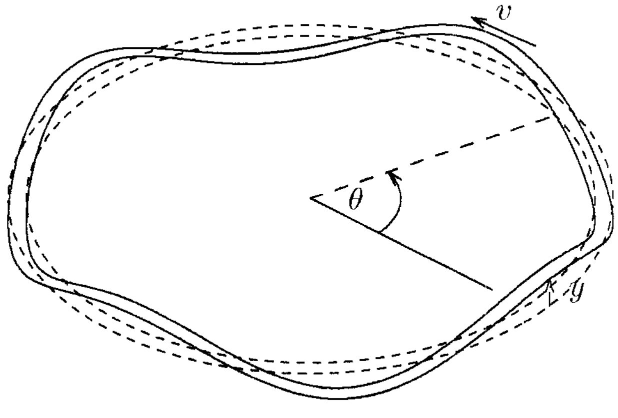
$$\Delta\omega_{\beta f} = (Q' - \eta_c(n_f + Q))\omega_0 \frac{\Delta p}{p} \ \ , \ \ \Delta\omega_{\beta s} = (Q' - \eta_c(n_s - Q))\omega_0 \frac{\Delta p}{p}$$

resulting in two frequency distributions $f(\omega_{\beta f})$, $f(\omega_{\beta s})$. We excite the beam by an acceleration $\hat{G}\exp(j\omega t)$ with frequency ω being close to $\omega_{\beta_f s}$ or $\omega_{\beta s}$ and get a velocity response of the center of mass of the distribution

$$\langle \hat{y} \rangle_f = -\frac{\hat{G}\omega}{2Q\omega_0} \int \frac{f(\omega_{\beta f}}{\omega_{\beta f} - \omega} d\omega_{\beta f} = -\frac{\hat{G}\omega}{2Q\omega_0} \left(\pi f(\omega) + jPV \int \frac{f(\omega_{\beta f})}{\omega_{\beta f} - \omega} \right) d\omega_{\beta f}.$$

$$\langle \hat{y} \rangle_s = \frac{\hat{G}\omega}{2Q\omega_0} \int \frac{f(\omega_{\beta s}}{\omega_{\beta s} - \omega} d\omega_{\beta s} = \frac{\hat{G}\omega}{2Q\omega_0} \left(\pi f(\omega) + jPV \int \frac{f(\omega_{\beta s})}{\omega_{\beta s} - \omega} \right) d\omega_{\beta s}.$$

The term $\pi f(\omega)$ is real, excitation and response are in phase resulting in an absorbtions of energy and damping, called Landau damping. It is only present if the excitation frequency ω is within the frequency distribution of the individual particles. The second term is imaginary and gives the out-of-phase response being of less interest.



The oscillating beam can induce a voltage in a transverse impedance which in turn applies a self acceleration G_s to the beam

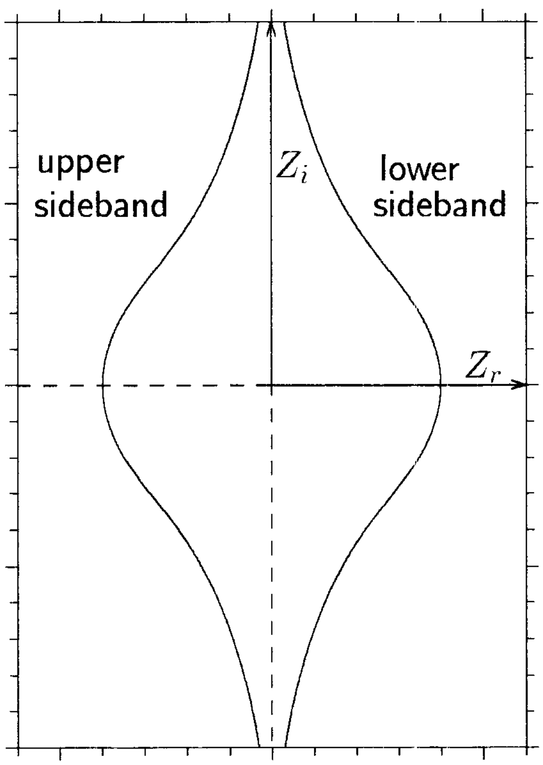
$$Z_T(\omega) = -\frac{\omega}{I\dot{y}(\omega)} \oint \left(\vec{E}(\omega) + [\vec{\beta} \times \vec{B}(\omega)]\right)_T ds \quad , \quad \hat{G}_s = -\frac{eZ_T I\langle\dot{y}\rangle}{\gamma m_0 2\pi R\omega}$$

If $\hat{G}_s = \hat{G}$ we can have a steady self sustained oscillation without external excitation, i.e. a threshold of an instability. Introducing this into the response we get for this threshold

$$1 = \frac{jecIZ_T(\omega)}{4\pi QE}\int \frac{f(\omega_{\beta s}}{\omega_{\beta f} - \omega} d\omega_{\beta f} = -\frac{ecIZ_T(\omega)}{4\pi QE}\left(\pi f(\omega) + jPV\int \frac{f(\omega_{\beta f})}{\omega_{\beta f} - \omega} d\omega_{\beta f}\right).$$

$$1 = \frac{-jecIZ_T(\omega)}{4\pi QE}\int \frac{f(\omega_{\beta s}}{\omega_{\beta s} - \omega} d\omega_{\beta s} = \frac{ecIZ_T(\omega)}{4\pi QE}\left(\pi f(\omega) + jPV\int \frac{f(\omega_{\beta s})}{\omega_{\beta s} - \omega} d\omega_{\beta s}\right).$$

These equations represent relations between the complex impedance and the complex beam response to an excitation. It is represented by the so-called stability diagram shown in the figure for a Gaussian distribution. If the impedances lies inside the central curve we have stability, outside an instability. The curve itself represents the threshold. Its shape is determined by the frequency distribution of the particles.

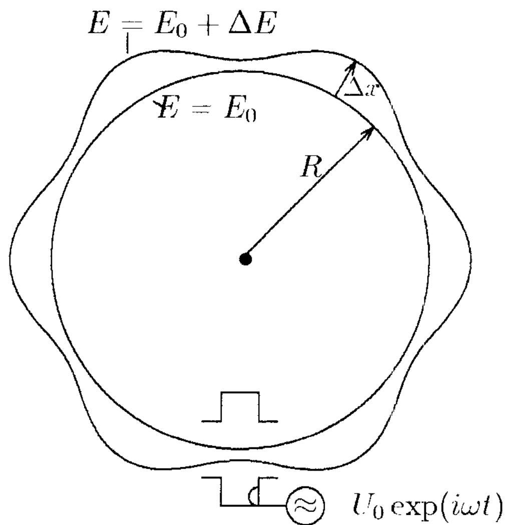


Longitudinal: The longitudinal dynamics of a coasting or unbunched beam is governed by the relation between the deviations in momentum and revolution frequency

$$\frac{\Delta E}{E} = \beta^2 \frac{\Delta p}{p} = -\frac{\beta^2}{\eta_c}\frac{\Delta \omega_0}{\omega_0}\ , \text{with}\ \ \eta_c = \alpha_c - \frac{1}{\gamma^2}.$$

The beam has an equilibrium energy distribution which translates into a distribution in revolution frequency

$$f_0(\Delta E) = \frac{1}{N}\frac{d^2 N}{d\theta dE} \ \rightarrow F_0(\Delta\omega_0) = \frac{1}{N}\frac{d^2 N}{d\theta d\omega_0}$$



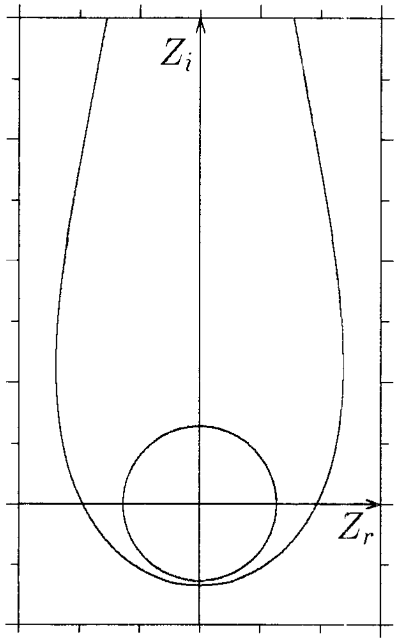
A stable beam has a continuous current I_0, however, exciting it with $U_0 \exp(j\omega t)$ close to $n\omega_0$ give a current perturbation

$$I_1(t) = \frac{-jNe^2\omega_0^3 U_0}{2\pi\beta^2 E)}\int \frac{dF_0(\omega_0)/dt}{\omega - n\omega_0}d\omega_0 = \frac{Ne^2\omega_0^3 U_0}{2\pi\beta^2 E)}\left(\pi\frac{dF_0}{d\omega_0}(\omega) - jPV\int\right).$$

This current I_1 can induce a voltage in an impedance Z. If it is as large or larger than U_0 it can replace the external excitation and keep the current modulation going or increase it. We get for this stability limit

$$1 = \frac{Ne^2\omega_0^3\eta Z(\omega)}{2\pi\beta^2 E}\left(\frac{\pi dF_0}{d\omega_0}(\omega) - jPV\int\frac{dF_0(\omega_0)/dt}{\omega - n\omega_0}d\omega_0\right).$$

This equation is a complex mapping which can be represented in form of a stability diagram which depends on the energy or revolution frequency distribution of the particles



$$1 = \frac{Ne^2\omega_0^3\eta Z(\omega)}{2\pi\beta^2 E}\left(\frac{\pi dF_0}{d\omega_0}(\omega) - jPV\int\frac{dF_0(\omega_0)/dt}{\omega - n\omega_0}d\omega_0\right).$$

To separate the dependence on the form of the distribution from the one on physical parameters like E, I_0, $\Delta p/p$ and η_c the stability diagram is normalized with the width the momentum spread. Taking many such diagrams and approximating them with a circle gives the (Keil-Schnell) stability criterion

$$\left|\frac{Z}{n}\right| \leq \frac{2\pi\beta^2 E\eta_c(\Delta p/p)^2}{eI_0}.$$

Important is the strong dependence on the momentum spread, or the connected frequency spread, which gives rise to Landau damping.

6) Bunch lengthening

The impedance of a ring consists often to a large part of many resonances with different frequencies ω_r, shunt impedance R_s and quality factors Q. At $\omega < \omega_r$ low frequencies their impedances are mainly inductive

$$Z(\omega) = R_s \frac{1 - jQ\frac{\omega^2-\omega_r^2}{\omega\omega_r}}{1 + \left(Q\frac{\omega^2-\omega_r^2}{\omega\omega_r}\right)^2} \approx j\frac{R_s\omega}{Q\omega_r} + ...$$

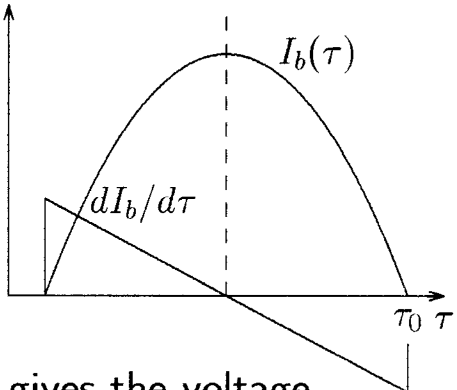
The sum impedance at low frequencies all these resonances divided by the mode number $n = \omega/\omega_0$ is called

$$\left|\frac{Z}{n}\right|_0 = \sum_k \frac{R_{sk}\omega_0}{Q_k\omega_{rk}} = L\omega_0.$$

with L being the inductance. A bunch with current $I_b(t)$ induces a voltage $V_i = -LdI_b/dt$ which is added to the RF-voltage

$$V(t) = \hat{V}\sin(h\omega_0 t) - L\frac{dI_b}{dt}.$$

Developing around t_s, using $\tau = t - t_s$, $\phi_s = h\omega_0 t_s$ and using a parabolic bunch



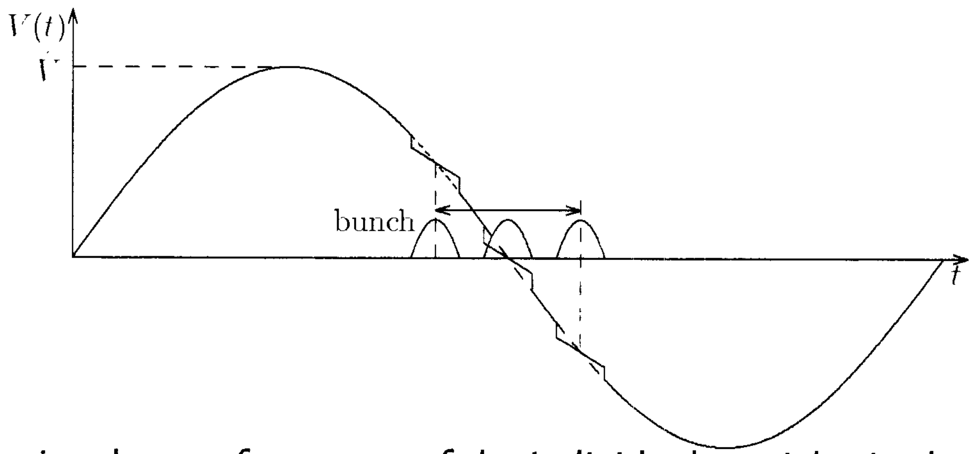
Average current I_0
$I_b(\tau) = \hat{I}(1 - \tau^2/\tau_0^2)$
$dI_b/d\tau = 3\pi I_0\tau/\tau_0^3$

gives the voltage

$$V = \hat{V}\sin\phi_s + \hat{V}\cos\phi_s h\omega_0\tau\left(1 + \frac{3\pi|Z/n|_0 I_0}{h\hat{V}\cos\phi_s(\omega_0\tau_0)^3}\right).$$

and synchrotron frequency shift for the particles in the bunch

$$\frac{\Delta\omega_s}{\omega_s} \approx \frac{3\pi|Z/n|_0 I_0}{2h\hat{V}\cos\phi_s(\omega_0\tau_0)^3}$$

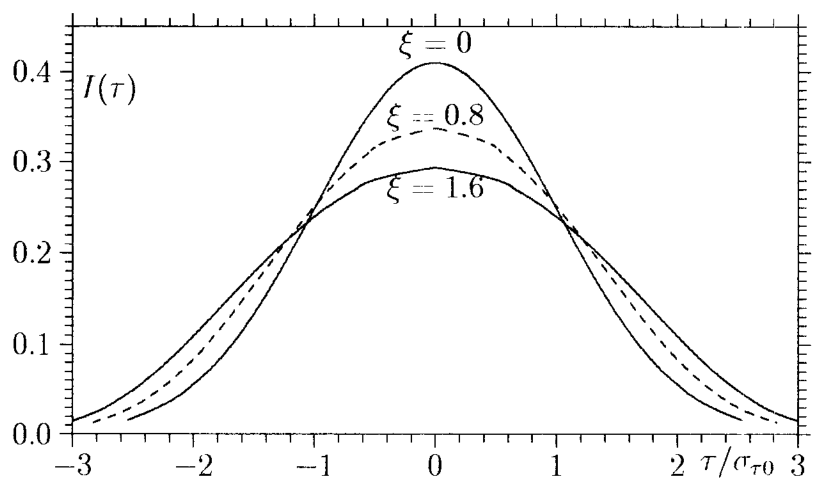


Only the incoherent frequency of the individual particles in the bunch is changed (reduced for $\gamma > \gamma_T$, increased for $\gamma < \gamma_T$). The coherent dipole (rigid bunch) is not affected by the inductive impedance. This can separate the coherent synchrotron frequency from the incoherent distribution and lead to a loss of Landau damping.

The reduction of the longitudinal focusing increase of the bunch length given by a 4th order equation for protons with constant phase space area

$$\left(\frac{\tau_0}{\tau_{00}}\right)^4 + \frac{3\pi|Z/n|_0 I_0}{h\hat{V}\cos\phi_s(\omega_0\tau_{00})^3}\left(\frac{\tau_0}{\tau_{00}}\right) - 1 = 0$$

The assumed parabolic bunch current is the projection of an elliptic phase space distribution. In this case the bunch form is not changed just its length increased. This is more complicated for other distribution like for the Gaussian shown in the figure.



$$\xi = \frac{\sqrt{2\pi}h^2 I_0|Z/n|_0}{\hat{V}\cos\phi(h\omega_0\sigma_{\tau 0})^3}$$